AUGUST, 1889.

ANNALS OF MATHEMATICS.

ORMOND STONE, Editor.
W. M. THORNTON, } Associate Editors.
R. S. WOODWARD, }

OFFICE OF PUBLICATION: UNIVERSITY OF VIRGINIA.

Volume 5, Number 1.

LIBRARY OF CONGRESS
195706
SEP 1 1889
SMITHSONIAN DEPOSIT

ALL COMMUNICATIONS should be addressed to ORMOND STONE, University of Virginia, Va., U. S. A.

Entered at the Post Office as second class mail matter.

ANNALS OF MATHEMATICS.

VOL. V. AUGUST, 1889. NO. 1.

ON THE ECCENTRICITY OF PLANE SECTIONS OF QUADRICS.

By PROF. H. B. NEWSON, Bushnell, Ill.

I.

CONJUGATE ECCENTRICITY OF A CONIC.

The foci of a conic, defined as the points of intersection of tangents to the conic from the two imaginary circular points at infinity, are four in number, two of them real and two imaginary. Salmon in his Conic Sections, chapter on Invariants and Covariants, gives a process of finding the co-ordinates of the foci of a conic from the equation. This method applied to the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, gives for the co-ordinates of the foci $(c, 0)$, $(-c, 0)$, $(0, ic)$, $(0, -ic)$, where $c = \sqrt{a^2 - b^2}$, and $i = \sqrt{-1}$. Thus it will be seen that the two real foci are situated on the transverse axis of the conic at equal distances from the centre, while the two imaginary foci consist of a pair of conjugate imaginary points situated on the conjugate axis, also equidistant from the centre.

Starting from this basis of fact, imaginary focal properties of conics may be developed which are counterparts of their real focal properties.* The method of development is parallel to that used by Salmon in proving the real focal properties.†

I shall call e_1 the conjugate eccentricity and proceed to find its relation to the ordinary eccentricity. Eliminating b^2 in the equations

$$e_1^2 = \frac{b^2 - a^2}{b^2} \quad \text{and} \quad e^2 = \frac{a^2 - b^2}{a^2},$$

and dividing by a^2, we have

$$e_1^2 = \frac{e^2}{e^2 - 1}; \quad \text{similarly,} \quad e^2 = \frac{e_1^2}{e_1^2 - 1}.$$

* [In fact it may be readily seen that, in problems relating to conics, the minor axis may be treated as though it were the major axis by merely interchanging a and b, and x and y. In this case e becomes e_1.—*O. S.*]

† [The details are omitted, since the process suggested in the preceding note is much simpler.—*O. S.*]

When $e_1 = 0$, the curve is a circle; when e_1 is imaginary, the curve is an ellipse; when $e_1 = \infty$, the curve is a parabola; when e_1 is real and greater than unity, the curve is an hyperbola; when e_1 is real and less than unity, the curve is such that the ordinary eccentricity is imaginary.

II.

TO FIND THE ECCENTRICITY OF ANY PLANE SECTION OF A QUADRIC SURFACE.

Let the triaxial ellipsoid, whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, be the type of the most general surface. By giving to the constants the proper signs and values we may obtain all the results wished.

Since the sections of a quadric by parallel planes are similar conics, we need to consider only central sections of the surface. The lengths of the axes of any central section of a quadric are given in Salmon's Geometry of Three Dimensions, Art. 101 (3d ed.), as follows: Let a' and b' be the semiaxes of the section, a, b, and c the semiaxes of the quadric, and α, β, γ the direction angles of the normal to the plane of section. We have then

$$\frac{1}{a'^2} + \frac{1}{b'^2} = \frac{\sin^2\alpha}{a^2} + \frac{\sin^2\beta}{b^2} + \frac{\sin^2\gamma}{c^2} = S. \tag{1}$$

$$\frac{1}{a'^2 b'^2} = \frac{\cos^2\alpha}{b^2 c^2} + \frac{\cos^2\beta}{a^2 c^2} + \frac{\cos^2\gamma}{a^2 b^2} = C. \tag{2}$$

Eliminating b'^2 in these equations and solving for a'^2, we find

$$a'^2 = \frac{S \pm \sqrt{S^2 - 4C}}{2C}. \tag{3}$$

Eliminating a' and solving for b' we have identically the same result. Hence we must take the upper sign before the radical for the one and the lower sign for the other. The upper sign will of course give the longer axis. Substituting these values in the formula, $\varepsilon^2 = 1 - \frac{b'^2}{a'^2}$, we have

$$\varepsilon^2 = \frac{2\sqrt{S^2 - 4C}}{S + \sqrt{S^2 - 4C}}. \tag{4}$$

In order to simplify this expression it is necessary to replace S and C by their values and reduce the quantity under the radical to the form of a perfect square. But this is generally impossible.

There are, however, some particular cases where the reduction is possible. Thus let one of the angles, say α, be zero; then, since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, $\cos^2\beta$ and $\cos^2\gamma$ are each equal to zero. In such a case the section is a principal section and its eccentricity is already known.

Again, if one of the angles, say β, is $90°$, then $\cos^2\alpha = \sin^2\gamma$ and $\sin^2\alpha = \cos^2\gamma$; whence we may write

$$S = \frac{\sin^2\alpha}{a^2} + \frac{1}{b^2} + \frac{\cos^2\alpha}{c^2}, \tag{5}$$

and

$$\sqrt{(S^2 - 4C)} = \pm\left(\frac{\sin^2\alpha}{a^2} - \frac{1}{b^2} + \frac{\cos^2\alpha}{c^2}\right). \tag{6}$$

Substituting in (4), employing the lower sign of the right hand member of (6), and writing ε_2 for ε, we have

$$\varepsilon_2^2 = 1 - \frac{b^2}{a^2}\sin^2\alpha - \frac{b^2}{c^2}\cos^2\alpha. \tag{7}$$

Employing the upper sign, and writing ε_2' for ε, we have

$$\varepsilon_2'^2 = \frac{1 - \frac{b^2}{a^2}\sin^2\alpha - \frac{b^2}{c^2}\cos^2\alpha}{-\frac{b^2}{a^2}\sin^2\alpha - \frac{b^2}{c^2}\cos^2\alpha} = \frac{\varepsilon_2^2}{\varepsilon_2^2 - 1}. \tag{8}$$

One of the values, therefore, of ε found by substituting (5) and (6) in (4) is the ordinary, and the other the conjugate, eccentricity.

Thus far we have made no assumption with regard to the relative lengths of the axes of the quadric. Let us suppose $a > b > c$; and put

$$\begin{aligned} \frac{a^2 - b^2}{a^2} &= e_1^2, & \frac{b^2 - a^2}{b^2} &= e_1'^2, \\ \frac{a^2 - c^2}{a^2} &= e_2^2, & \frac{c^2 - a^2}{c^2} &= e_2'^2, \\ \frac{b^2 - c^2}{b^2} &= e_3^2, & \frac{c^2 - b^2}{c^2} &= e_3'^2; \end{aligned} \tag{9}$$

in which the accents mark the conjugate eccentricities of the principal sections of the quadric. Substituting in (7), we find

$$\varepsilon_2^2 = e_1^2\sin^2\alpha + e_3'^2\cos^2\alpha. \tag{10}$$

Similarly, when $\gamma = 90°$,

$$\varepsilon_1^2 = 1 - \frac{c^2}{b^2}\sin^2\beta - \frac{c^2}{a^2}\cos^2\beta$$

$$= e_3^2 \sin^2\beta + e_2^2 \cos^2\beta, \tag{11}$$

$$\varepsilon_1'^2 = \frac{\varepsilon_1^2}{\varepsilon_1^2 - 1};$$

and when $\alpha = 90°$,

$$\varepsilon_3^2 = 1 - \frac{a^2}{c^2}\sin^2\gamma - \frac{a^2}{b^2}\cos^2\gamma$$

$$= e_3'^2 \sin^2\gamma + e_1'^2 \cos^2\gamma, \tag{12}$$

$$\varepsilon_3'^2 = \frac{\varepsilon_3^2}{\varepsilon_3^2 - 1}.$$

It is evident in each of these cases that the cutting plane is perpendicular to one of the principal planes of the quadric.

Thus far we have said nothing as to which of the values of ε^2 represent the ordinary and which the conjugate eccentricity. The choice of values depends upon the sign. In the case of the ellipse one of these values is always positive and the other always negative. In equation (11) e_2^2 and e_3^2 are both positive, and hence ε_1^2 is always positive and the square of the ordinary eccentricity. In (10) e_1^2 is positive and $e_3'^2$ is negative, and hence ε_2^2 will be sometimes positive and sometimes negative, depending on the value of α. In (12) $e_1'^2$ and $e_2'^2$ are both negative, and hence ε_3^2 is always negative, and, in accordance with the above notation, must be written

$$\varepsilon_3'^2 = e_2'^2 \sin^2\gamma + e_1'^2 \cos^2\gamma.$$

Again, since $\sin^2\alpha = \cos^2\gamma$ when $\beta = 90°$, it is evident that (10) may be written

$$\varepsilon_2^2 = e_1^2 \cos^2\gamma + e_3'^2 \sin^2\gamma.$$

Similarly, (11) and (12) may be written

$$\varepsilon_1^2 = e_2^2 \cos^2\alpha + e_3^2 \sin^2\alpha,$$

$$\varepsilon_3'^2 = e_2'^2 \cos^2\beta + e_3'^2 \sin^2\beta.$$

Thus far we have restricted the investigation to the ellipsoid and found for ε_1^2, ε_2^2, ε_3^2 values which depend only upon a variable angle and the eccentricities of the principal sections. Hence in the above equations e_1, e_2, and e_3 are limited to values between 0 and 1. But if the ellipsoid by a variation of its axes becomes a paraboloid or hyperboloid, one or more of these quantities pass to unity or

beyond. Hence in order to extend the application of ε_1, ε_2, and ε_3 to the general surface of the second degree we have only to consider e_1, e_2, and e_3 as capable of assuming all values between 0 and ∞.

III.

A SECOND METHOD OF FINDING ε.

There is another and indeed a much shorter method of finding ε_1, ε_2, and ε_3, but it is applicable only when the cutting plane is perpendicular to one of the principal planes of the quadric, and not to the general case.

Take the ellipsoid whose equation is $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, and suppose a cutting plane perpendicular to the plane of xy and passing through the centre. The equation of the principal section by the plane of xy is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$; or, in polar co-ordinates, the centre being the pole and the axis of x the prime radius,

$$R_1^2 = \frac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta}. \tag{1}$$

Now R_1 is the semimajor axis of a section by a plane through the centre perpendicular to the plane of xy, and making an angle θ with the axis of x. c is the semiminor axis of the section. Hence we have readily

$$\begin{aligned} \varepsilon_1^2 &= \frac{R_1^2 - c^2}{R_1^2} \\ &= 1 - \frac{c^2}{b^2}\sin^2\theta - \frac{c^2}{a^2}\cos^2\theta \qquad (2) \\ &= e_2^2 \cos^2\theta + e_3^2 \sin^2\theta. \end{aligned}$$

In the same manner, we find in the plane of xz that if

$$R_2^2 = \frac{a^2 c^2}{a^2 \sin^2\theta + c^2 \cos^2\theta},$$

then

$$\begin{aligned} \varepsilon_2^2 &= 1 - \frac{b^2}{R_2^2} \\ &= 1 - \frac{b^2}{c^2}\sin^2\theta - \frac{b^2}{a^2}\cos^2\theta \\ &= e_1^2 \cos^2\theta + e_3'^2 \sin^2\theta. \end{aligned}$$

Since R_2^2 is sometimes greater and sometimes less than b^2, ε_2^2 is sometimes positive and sometimes negative.

In the yz plane a^2 is always greater than the radius vector; whence, if

$$R_3^2 = \frac{b^2c^2}{b^2\sin^2\theta + c^2\cos^2\theta},$$

then

$$\begin{aligned}\varepsilon_3'^2 &= 1 - \frac{a^2}{R_3^2} \\ &= 1 - \frac{a^2}{c^2}\sin^2\theta - \frac{a^2}{b^2}\cos^2\theta \\ &= e_1'^2\cos^2\theta + e_3'^2\sin^2\theta.\end{aligned}$$

IV.

ε_1, ε_2, AND ε_3 FOR DIFFERENT VARIETIES OF QUADRICS.

The results obtained in the last section are the normal forms for the ellipsoid when $a^2 > b^2 > c^2$. If c^2 is negative, or both b^2 and c^2, we have the hyperboloid of one or of two sheets. The *form* of ε_1^2, ε_2^2, and ε_3^2 in either case remains unchanged. In the elliptic hyperboloid of one sheet, for which c^2 is negative, we find e_1^2 less than unity, and e_2^2 and e_3^2 both greater than unity. Since these quantities are independent of the absolute values of the axes, they still represent the eccentricities when the hyperboloid degenerates into its asymptotic cone. Hence, for such a cone the form of ε_1^2, ε_2^2, and ε_3^2 remains unchanged.

If $a^2 = \infty$, the other axes remaining the same, the surface is an elliptic paraboloid. e_1^2 and e_2^2 are both equal to unity, and hence $e_1'^2$ and $e_2'^2$ are both equal to infinity.

If $b^2 = c^2$, the surface is one of revolution around the axis of x. $e_1^2 = e_2^2$ and $e_3^2 = 0$. Hence (2), (3), and (4) of last section become

$$\varepsilon_1^2 = \varepsilon_2^2 = e_1^2\cos^2\theta \quad \text{and} \quad \varepsilon_3'^2 = e_1'^2. \tag{1}$$

If $a^2 = b^2$, the surface is one of revolution around the axis of z, and we have

$$\varepsilon_2^2 = \varepsilon_3^2 = e_3'^2\sin^2\theta \quad \text{and} \quad \varepsilon_1^2 = e_3^2. \tag{2}$$

If in this case c^2 is greater than a^2, $e_3'^2$ is of course real. This last formula remains true when the hyperboloid of revolution degenerates into its asymptotic cone, which is a right circular cone.

The meaning of $e_3'^2$ in this case deserves special consideration. $e_3'^2 = \frac{c^2 + b^2}{c^2}$, where c represents a certain distance from the apex of the cone measured on the axis of z. We may so choose the base of the cone that c shall represent the distance from the apex to the centre of the base. Then $b = a$ is the radius of

the base. Let φ be the angle which any element of the cone makes with the plane of the base. We have then a right triangle in which $c^2 + b^2$ is the square on the hypothenuse and c^2 the square on the side opposite φ. Hence

$$\frac{c^2 + b^2}{c^2} = c_3'^2 = \frac{1}{\sin^2\varphi};$$

$$\therefore \quad \varepsilon^2 = \frac{\sin^2\theta}{\sin^2\varphi}. \tag{3}$$

If $\varphi = 90°$, the cone becomes a cylinder and $\varepsilon^2 = \sin^2\theta$. This last result follows immediately from the geometry of the cylinder, since the semiminor axis of a section is always equal to the radius of the cylinder, and the semimajor axis is $b \sec\theta$. Hence

$$\varepsilon^2 = \frac{b^2 \sec^2\theta - b^2}{b^2 \sec^2\theta} = \sin^2\theta.$$

V.

A THIRD METHOD OF FINDING ε.

There is a third method of finding ε, which may be applied separately to each variety of quadric. I will apply it to the case of the right circular cone.

The equation of a cone, asymptotic to an hyperboloid of revolution, the origin being at the vertex, is

$$\frac{x^2 + y^2}{a^2} - \frac{z^2}{c^2} = 0. \tag{1}$$

To transform this to a new origin on the axis of z at a distance d below the vertex, we write $z - d$ for z; whence

$$(x^2 + y^2)\frac{c^2}{a^2} = (z - d)^2. \tag{2}$$

Now $\frac{c}{a}$ is the tangent of the angle which an element of the cone makes with the base. Calling this angle φ, we have

$$(x^2 + y^2)\tan^2\varphi = (z - d)^2. \tag{3}$$

Let the cone be cut by any plane

$$z = x \tan\theta, \tag{4}$$

passing through the axis of y, where θ is the angle which the cutting plane makes with the base. Eliminating z between (3) and (4), we have

$$(x^2 + y^2)\tan^2\varphi = (x\tan\theta - d)^2, \tag{5}$$

which is the equation of the projection of the curve of intersection on the plane of xy. The plane of the curve meets the plane of xy in the axis of y. To obtain the equation of the curve itself we have only to replace x by $x\cos\theta$. Substituting and expanding,*

$$x^2(\tan^2\varphi - \tan^2\theta)\cos^2\theta + y^2\tan^2\varphi + 2dx\sin\theta = d^2, \tag{6}$$

which is the equation of a conic. Since this equation contains no term in xy, we know that the coefficients of x^2 and y^2 are the squares of the semiaxes. Substituting these coefficients in the formula for ε^2, we have

$$\varepsilon^2 = \frac{a^2 - b^2}{a^2} = \frac{\sin^2\theta}{\sin^2\varphi}. \tag{7}$$

When θ and φ both lie between 0° and 90° it is evident that ε is less, equal to, or greater than unity according as θ is less, equal to, or greater than φ; and the curve is respectively an ellipse, parabola, or hyperbola. $\varepsilon = 0$ when $\theta = 0$, and the curve is a circle.

By a process exactly similar to the above we might deduce the expression for ε for every variety of surface of second degree.

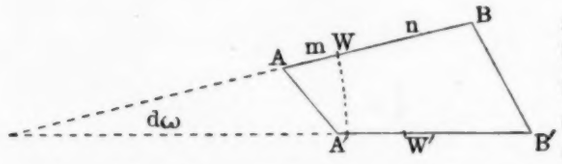
The foregoing theory of the eccentricity of plane sections of quadrics is not only interesting in itself and capable of yielding new results, but leads by direct and easy paths to many old theorems which have hitherto been reached only by roundabout processes.

*This equation is given in many elementary text-books. See Wentworth, Ray, etc.

THE PROBLEM OF THE POLAR PLANIMETER.

By Prof. W. H. Echols, Rolla, Mo.

I have found it interesting to consider the general problem of the planimeter as that of finding the area swept over by a right line of constant length moving in a plane, and having an attachment for recording the normal movement of a point on it. It appears to be more simple than the demonstrations usually given in the text books for the action of the instrument in special cases.



Let AB be the line of constant length l, and let the wheel be on the line at a distance m from A, and n from B, and at a distance c from the middle of AB.

Let AB make a small shift to $A'B'$, turning through the angle $d\omega$. Let ρ be the distance of A from the point of contact of the line with its envelope.

The wheel record is

$$dW = (\rho + m)d\omega;$$

whence

$$l\rho d\omega = ldW - lmd\omega.$$

The elementary area swept over by the line is

$$\begin{aligned} d\Omega &= \tfrac{1}{2}(\rho + l)^2 d\omega - \tfrac{1}{2}\rho^2 d\omega, \\ &= ldW + \tfrac{1}{2}l(l - 2m)d\omega, \\ &= ldW + \tfrac{1}{2}(n^2 - m^2)d\omega, \\ &= ldW + cld\omega. \end{aligned}$$

$$\Omega = lW + cl \cdot \int d\omega$$

is the general equation of the planimeter, giving the area between the curves (A) and (B) described by the extremities of l, and the initial and final positions of l.

All particular cases which can exist are determined by the limits of $\int d\omega$ and the direction in which the line turns. If we consider the wheel record to increase and the line to move through positive angles as it turns around A as a centre in the direction of clock hands, then for this motion both dW and $d\omega$ are positive; for the opposite motion about A they are both negative. If l turns about B clockwise, dW is negative and $d\omega$ positive, and their signs are changed when the direction of motion is reversed.

If the extremities describe closed curves of areas (A) and (B), and the area of one is known, that of the other is determined.

1. If the line does not revolve, but oscillates back to its first position, then $\int d\omega = 0$ always, and

$$\Omega = (A) \sim (B) = lW.$$

If A or B moves in a line of any kind and retraces its path while the other describes a closed curve, then the area of that curve is lW.

In particular, if the line traveled by the oscillating point be straight or circular, we have the rolling planimeter whose roller is a cylinder or frustum of a cone of revolution.

It is therefore not necessary, so far as accuracy is concerned, that the two rollers should be segments of the same cylinder in the construction of the actual instrument.

2. If the line does revolve once then the formula for the area between the curves is

$$\Omega = (A) \sim (B) = lW + 2\pi cl,$$

in which the signs of the two terms are determined as above. The sign of W, however, always takes care of itself, as shown by the wheel readings.

If the curve (B) envelops the curve (A) the signs of both terms are the same; and if (A) envelops (B) they have different signs for position of roller nearer A than B.

If the path of one of the points crosses that of the other, then the formula for the area between the curves gives the difference between the sum of the areas which lie on one side of one of the curves and the sum of those which lie on the other side of it. In which case if the area of this curve be added (with proper sign) to this difference the area of the other closed curve is determined.

The area of one of the curves may be zero, in which case the area of the other is given, and we may have a constant to add for the rolling planimeter; but here nothing is gained by complete revolution, and the mechanism of the instrument prevents it.

In particular, if the point A moves on the circumference of a circle of radius r,

$$\Omega = lW + \pi(n^2 - m^2),$$

$$(B) = lW + \pi(n^2 - m^2) + (A),$$

$$= lW + \pi(l^2 + r^2 + 2ml),$$

and we have the polar planimeter as usually constructed where $n = l + m$, the second term being *the constant* for the instrument and the first the registration of the roller.

If $c = 0$, or the wheel is in the middle of the bar, then

$$d\Omega = l dW,$$

and the wheel record is the area between the curves always. Which agrees with the Properties of Guldinus.

If the wheel be attached to the bar by an offset of length b which is distant m and n from A and B respectively, the formulæ are identical with those above; for if the shift of AB is small and finite the wheel record is

$$\Delta W = (\rho + m)\Delta\omega + b(1 - \cos \Delta\omega),$$

the last term of which disappears in the limit. The normal movements of all points in the same perpendicular to a line are the same. Hence, if the plane of the wheel be normal to the line at its middle point, the area swept over is always lW, the wheel record, and no constant is to be added for revolution.

The formula for the area swept over by a right line moving in a plane in terms of the normal movement of a point on it furnishes an elegant proof of Holditch's Theorem.

Let P be a point on the line AB, of variable length λ, dividing AB in the constant ratio a/b.

If dW be the normal movement of the middle point C of AB, the elementary area swept over by AB is

$$d\Omega = \lambda dW.$$

That swept over by AP is

$$d\Omega' = \frac{a}{a+b}\lambda dW + \tfrac{1}{2}(\overline{PC}^2 - \overline{AC}^2)d\omega,$$

$$= \frac{a}{a+b}d\Omega - \frac{ab}{(a+b)^2} \cdot \tfrac{1}{2}\lambda^2 d\omega.$$

Integrating for a complete circuit and writing (O) for the area of the closed curve $\lambda = f(\omega)$ we have

$$(P) - (A) = \frac{a}{a+b}[(B) - (A)] - \frac{ab}{(a+b)^2}(O),$$

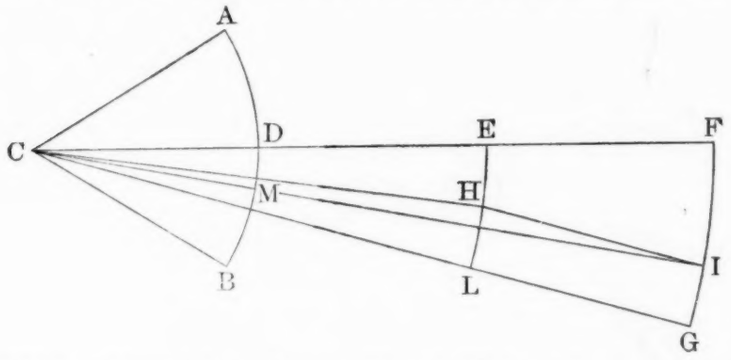
or

$$(P) = \frac{a(B) + b(A)}{a+b} - \frac{ab}{(a+b)^2}(O).$$

THE APPROXIMATE INSCRIPTION OF CERTAIN REGULAR POLYGONS.

By Prof. H. A. Howe, Denver, Col.

1. *To inscribe a regular nonagon in a circle.*—Let ACB be a central angle of 60° in the given circle. Bisect ACB by CF, laying off DE and EF equal to CD.



Bisect FCB by CG, and draw the arcs EL and FG from C as a centre. Bisect FCG, thus determining H, the mid-point of EL. From H draw HI parallel to CG, and from the point I where it cuts FG draw IC. Then will ICG be nearly equal to one third of FCG, or 5°. By solving the triangle HIC we find that $CIH = ICG = 4°\ 59'\ 31''.37$. Hence $ACM = 40°\ 0'\ 28''.63$, and a chord drawn from A to M will be very nearly equal to the side of a regular nonagon inscribed in the circle whose radius is $AC : 9 \times 28''.63 = 257''.67$, which is $\frac{1}{5030}$ of the entire circumference. If one half the angle FCI were subtracted from an angle of 45°, an angle of $39°\ 59'\ 45''.68$ would result, which is a closer approximation to 40°.

It is evident that there are many ways of getting angles more nearly equal to 5°; we give a few, each of which depends upon the trisection of a small angle, together with successive bisections of angles; thus:

$$5° = \tfrac{15}{2}° - \tfrac{1}{3}\cdot\tfrac{15}{2}° \qquad (A)$$
$$= \tfrac{15}{4}° + \tfrac{1}{3}\cdot\tfrac{15}{4}° \qquad (B)$$
$$= \tfrac{15}{4}° + \tfrac{15}{8}° - \tfrac{1}{3}\cdot\tfrac{15}{8}° \qquad (C)$$
$$= \tfrac{15}{4}° + \tfrac{15}{16}° + \tfrac{1}{3}\cdot\tfrac{15}{16}° \qquad (D)$$
$$= \tfrac{15}{4}° + \tfrac{15}{16}° + \tfrac{15}{32}° - \tfrac{1}{3}\cdot\tfrac{15}{32}° \qquad (E)$$
$$= \tfrac{15}{4}° + \tfrac{15}{16}° + \tfrac{15}{64}° + \tfrac{1}{3}\cdot\tfrac{15}{64}°, \qquad (F)$$
. . . . . .

If the trisections included in solutions $(A), \ldots (F)$ be performed in the same manner in which FCG was trisected (getting ICG as the approximate value of $\frac{1}{3}FCG$), it is easy to compute the error of each solution in the following way:—

Let FCG now denote any one of these angles to be trisected; denote ICG by x and HCG by y; let $2F(y)$ denote $y - \sin y$, and $2F(x)$ denote $x - \sin x$.

Since $\sin y = \frac{3}{2} \sin x$, we have

$$y - 2F(y) = \tfrac{3}{2}\, x - \tfrac{3}{2} . 2F(x). \qquad (1)$$

By development,

$$2F(y) = \tfrac{1}{6} \sin^3 y + \tfrac{3}{40} \sin^5 y + \tfrac{5}{112} \sin^7 y + \tfrac{35}{1152} \sin^9 y + \ldots . \qquad (2)$$

Likewise, substituting $\frac{2}{3} \sin y$ for $\sin x$, we get

$$2F(x) = \tfrac{4}{81} \sin^3 y + \tfrac{4}{405} \sin^5 y + \tfrac{40}{15309} \sin^7 y + \tfrac{140}{177147} \sin^9 y + \ldots . \qquad (3)$$

Hence

$$\frac{2F(x)}{2F(y)} = \tfrac{8}{27} - (\tfrac{2}{27} \sin^2 y + \tfrac{154}{51030} \sin^4 y + \tfrac{6518}{4133430} \sin^6 y + \ldots) = \tfrac{8}{27} - \varphi(y), \quad (4)$$

where $\varphi(y)$ denotes the series in the ( ).

Find the value of $F(x)$ from (4), substitute it in (1), and reduce, obtaining

$$y - \tfrac{3}{2}\, x = \tfrac{5}{9} . 2F(y) + \tfrac{3}{2}\, \varphi(y) . 2F(y),$$

or

$$\tfrac{2}{3}\, y - x = \tfrac{10}{27} . 2F(y) + \varphi(y) . 2F(y). \qquad (5)$$

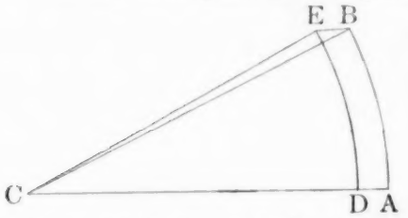
Now $\frac{2}{3}\, y - x$ is the error of x, and (5) furnishes an easy method of computing this error. Thus we find the following table:—

Solution.	$2y$	Error of x.	$\frac{9 \times \text{error of } x}{1{,}296{,}000''}$
	°	″	
(A)	7·5	3·573	1 : 40,290±
(B)	3·75	0.4462	1 : 322,700±
(C)	1.875	0.0558	1 : 2,583,000±
(D)	0.9375	0.00698	1 : 20,660,000±
(E)	0.46875	0.00087	1 : 165,000,000±
(F)	0.234375	0.00011	1 : 1,300,000,000±

It thus becomes apparent that it is possible to find a third of any acute angle without much labor, and attain a result the theoretical error of which is far less than the unavoidable errors of the most careful draughtsmen with the best drawing instruments manufactured. When the angle is between 45° and

90°, it is best to find one third of its complement, and subtract it from 30°. If the given angle is obtuse it is best to find a third of its supplement, and subtract it from 60°.

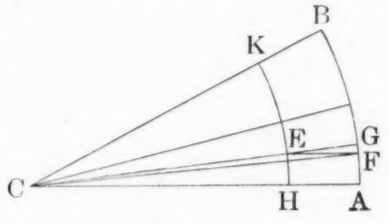
2. *The Inscription of a Regular Polygon of Eleven Sides.— 1st Method.—* Divide the radius CA of the given circle into twelve equal parts, of which CD contains eleven. Construct $ACB = 30°$. Draw AB and DE. From B draw BE parallel to AC. Draw CE. Then $ECD = 33° 3' 20''.+$, which is one per cent. larger than 360° : 11. If the arc ED had been drawn with a radius of 11.1, ECD would have been equal to 32° 43′ 13″.60, which is $\frac{1}{4793}$ smaller than 360° : 11.



2d Method.— If a triangle be formed of sides 6, 10, 11, the angle opposite 6 will be 32° 45′ 50″, which is $\frac{1}{897}$ larger than 360° : 11.

3. *The Inscription of a Regular Polygon of Thirteen Sides.*— In the preceding figure let $CD = 12$, and $CA = 13$, and the angle $ECD = 30°$, EB being parallel to AC. Then $ACB = 27° 29' 11''.14$, which is $\frac{10}{1345}$ smaller than 360° : 13. If CA be taken as $12\frac{10}{11}$, ACB will be 15″.27 larger than desired, or $\frac{1}{6529}$ too large.

4. *The Multisection of Small Angles.*— If we wish to find one fifth of ACB, by successive bisections we obtain $ACG = \frac{1}{4} BCA$. Lay off $CH = 4$ and $CA = 5$, and draw the arcs BA and HK. From E, where CG cuts HK, draw EF parallel to CA. FCA is nearly one fifth of BCA. Denote GCA by a, and FCA by x. By reasoning similar to that employed before we may show that



$$\tfrac{4}{5}a - x = (\tfrac{36}{125}. + \tfrac{1296}{15625} \sin^2 a + \ldots) 2F(a).$$

If $BCA = 15°$, FCA is 2° 59′ 57″.22, which is 2″.78 too small.

To find $\frac{1}{7}$ of BCA. By successive bisections, find $FCA = \frac{1}{8} BCA$. Lay off $CA = 8$ and $CH = 7$, and draw BA and KH; also draw FE parallel to CA. ECA is nearly equal to $\frac{1}{7} BCA$. Denoting FCA by a, and ECA by x, we find

$$\tfrac{8}{7}a - x = (-\tfrac{120}{343} - \tfrac{3456}{16807} \sin^2 a - \ldots) 2F(a).$$

If $BCA = 15°$, $ECA = 2° 8' 34''.71$, which is 0″.42 too large.

By combinations of the methods which have been set forth one may easily find $\frac{1}{9}$, $\frac{1}{10}$, $\frac{1}{11}$, etc. of any small angle with considerable accuracy. These methods are so simple that it is quite probable that they are not new; but I have never heard of them.

NOTE ON PERSPECTIVE PROJECTION.

By Prof. W. H. Echols, Rolla, Mo.

In order to give verisimilitude to the picture the centre of projection must be at considerable distance from the original and the picture plane; this gives rise to the necessity for drawing long lines through distant points by the use of either the plans of the rays directly, or the employment of *diagonals* and their vanishing point. In the latter case, the difficulty is somewhat alleviated by means of *reduced distances;* in other words, by using *diagonals* which make a large angle with the picture plane, and whose vanishing point is correspondingly near the centre of the picture. The objection to this last method is that in order to locate the trace of the diagonal with the perspective plane a subdivision of the distance of the point in space from the picture plane must be used, which necessitates an additional construction for each point, involving time and a certain amount of inaccurary due to transferring distances. Where the plans of the rays concurring in the distant plan of the centre of projection are used, the centro-linead is inaccurate, and the use of a harmonic range with the quadrilateral construction too tedious.

It is believed that the following construction obviates some of the difficulties met with in drawing such projections by the methods now in use.

Let the orthogonal plan of the original lie in any horizontal plane α; select any straight line in this plane as the trace with it of the vertical picture plane β. Let the centre of projection be any point in space, and from it project the orthogonal plan upon the plane β, into what will be called the perspective plan. Rebat β into α. The centre of projection comes down into α by describing a quadrant of a circle whose plane is normal to α and β, and whose centre is the orthogonal plan of the centre of projection. The plan and its perspective are now superposed into two homological figures in a plane, the trace of α with β being the axis, and the development of the centre of projection the centre of homology. If the perspective of any one point of the plan be known, the whole perspective plan can at once be drawn (Cremona, Proj. Geom., Art. 22, *et seq.*).

It is obvious that the position of the centre of projection may always be selected, without loss of advantageous position, so that its development into α may have any position we choose in that plane with respect to the plan. The construction of the perspective plan may thus be confined to the desired area of drawing surface.

In particular, let the centre of projection be equidistant from α and β, in front of β and above α, the orthogonal plan being also in front of β. After the

rebatment of β back into α the centre is found at O, the intersection of the axis with the plan r of the central ray, and the perspective of any point P on r comes up into α on r to a point P' homological with P, the distance OP is a mean proportional between PP' and PC', where C' is the orthogonal plan of the centre of projection. In other words, if P be selected so that its distance from O is one nth of its distance from C', then PP' is one nth of OP. This fixes a pair of homological points necessary to complete the construction of the perspective plan.

It is well to let the orthogonal plan touch the axis; this condenses the drawing, and literally superposes the perspective plan on the orthogonal; each perspective point is near its original and easily identified.

Points in the picture lie on normals to the axis through points of the perspective plan, where they are located by means of the perspective of *perpendiculars* which vanish in the centre of the picture, and pass through their traces with β. These traces are referred to the *prime vertical* and the *horizon* as axes. They lie on normals to the axis of homology through points in the orthogonal plan, where they are located by the distance of the original point from the plane of the horizon, as determined by a given orthogonal elevation, or equivalent. These latter are the only measures taken, or which have to be transferred, with dividers. This method gives oblique intersections near the vertical through the centre of the picture. The diagonal may be used to good advantage in that case, the plan of any ray serving as the direction for a diagonal which locates the vanishing point; otherwise the vanishing point may be assumed at pleasure and the plan of the ray passing through it constructed as above, and perspectives of particular diagonals constructed by the usual methods.

ON THE EXPRESSION FOR THE HESSIAN OF A BINARY QUANTIC IN TERMS OF THE ROOTS.

By Mr. James McMahon, Ithaca, N. Y.

In Salmon's Higher Algebra (p. 185) is given an expression for the Hessian of a binary cubic in terms of the roots, viz:—

$$-18\,H = a_0^2 \Sigma (\alpha - \beta)^2 (x - \gamma y)^2,$$

which is derived by a special method from the harmonic relations between the roots of the cubic and of the Hessian. On page 192 is given, without proof, for the Hessian of a quartic the expression

$$-48\,H = a_0^2 \Sigma (\alpha - \beta)^2 (x - \gamma y)^2 (x - \delta y)^2;$$

and on page 259 for the Hessian of a quintic the expression

$$-100\,H = a_0^2 \Sigma (\alpha - \beta)^2 (x - \gamma y)^2 (x - \delta y)^2 (x - \varepsilon y)^2.$$

I propose to prove that this form is general, and that the expression for the Hessian of the binary n-ic, in terms of the roots, is

$$-n^2(n-1)\,H = a_0^2 \Sigma (\alpha_1 - \alpha_2)^2 (x - \alpha_3 y)^2 (x - \alpha_4 y)^2 \ldots (x - \alpha_n y)^2;$$

for this expression is evidently from its form a covariant (p. 125), and of the degree $2(n-2)$ in x, y, and of the order 2 in the coefficients of the quantic (p. 59); again, the covariant $\frac{d^2u}{dx^2}\cdot\frac{d^2u}{dy^2} - \left(\frac{d^2u}{dxdy}\right)^2$, which is the Hessian of the quantic u (with a numerical factor), is evidently of the same degree and order as the above covariant; hence, to show that these two covariants differ only by a factor (involving n alone), we must show that there can be but *one* covariant of the degree $2(n-2)$ in x, y, and of the order 2 in the coefficients. Now, it is seen from page 134 that the source of either of these covariants has the weight $\frac{1}{2}(n\theta - p)$, which $= 2$, when $\theta = 2$ and $p = 2(n-2)$; hence the source of either covariant is of the form $a_0 a_2 + \lambda a_1^2$, and, in order that this may be reduced to zero by the operator $a_0 \frac{d}{da_1} + 2a_1 \frac{d}{da_2} + \ldots$, we must have $\lambda = -1$, therefore there is but one source, and but one covariant of this degree and order.

To determine the multiplier, take the special binary n-ic

$$a_0x^n + na_1x^{n-1}y,$$

wherein one root is $-\frac{na_1}{a_0}$ and the other $n-1$ roots are zero, and compare the coefficients of $x^{2(n-2)}$ in the two expressions for the covariant: the coefficient of $x^{2(n-2)}$ in the Hessian is $a_0a_2 - a_1^2$, which in this case becomes $-a_1^2$; and the coefficient of $x^{2(n-2)}$ in

$$\Sigma(a_1 - a_2)^2(x - a_3y)^2(x - a_4y)^2 \ldots (x - a_ny)^2$$

is $\Sigma(a_1 - a_2)^2$, which becomes $(n-1)\left\{\frac{na_1}{a_0}\right\}^2$; hence the multiplier is determined, and we have the identity

$$-n^2(n-1)H = a_0^2\Sigma(a_1 - a_2)^2(x - a_3y)^2(x - a_4y)^2 \ldots (x - a_ny)^2.$$

NOTE ON SYMBOLS.

By Prof. A. Hall, Washington, D. C.

I am glad that Professor Oliver has undertaken the discussion of mathematical notation, and I wish to call attention to the need of something being done in a negative direction.

(α). In the case of the inverse trigonometric functions our text books universally use the illogical expressions $\sin^{-1}x$, $\cos^{-1}x$, etc. This notation has the sanction of long usage in England and in this country, but it would be much better, I think, to follow the example of French authors and write arc sin x, arc cos x, etc.

(β). In writing the differential coefficient of a function our text books show a confusing variety of notation. We have the notation of Leibnitz, $\frac{du}{dx}$; the Newtonian symbol $\dot{u}$; and the complex symbol $D_x u$. Of these it seems to me that the first is the most expressive and the best, and it is generally adopted by European mathematicians. The confusion is increased by the use of the capital D to denote sometimes a partial and sometimes a complete derivative. The partial derivative is now well expressed after the manner of Jacobi by $\frac{\partial u}{\partial x}$.

(γ). In a special memoir an author can of course use such notation as he pleases, but text books should be designed to give the student clear and firm ideas of the principles and symbols he uses. For a beginner the chief difficulty in the application of mathematics does not lie in the solution of algebraic and differential equations, nor in integration, since in many cases this part of the work is much like taking a logarithm from the tables; but the most serious part of the work lies in the correct expression of ideas and in the formation of the right equations. For these reasons it seems to me that authors of text books should endeavor to select the simplest and most expressive notation. And it is for reasons like these that I think the method of infinitesimals will never be dispensed with, since it is so closely connected with geometrical conceptions.

We have the statement of one of the most eminent of living mathematicians, who began life as a lawyer, that while the law seems designed to confuse and complicate matters, the end of mathematics is to make things plain. I hope our mathematical writers will give heed to this opinion.

SOLUTIONS OF EXERCISES.

ACKNOWLEDGMENTS.

R. J. Adcock 232, 233; Charlotte C. Barnum 229, 234, 235; T. M. Blakslee 232, 233; Geo. W. Coakley 244; H. W. Draughon 229, 232, 233, 234, 235; E. Frisby 229, 230, 234, 236, 237; R. H. Graves 235, 238, 239, 241, 244; R. A. Harris 218, 229, 232, 233; J. E. Hendricks 208; L. M. Hoskins 245; Artemas Martin 208, 215, 216, 218; James McMahon 231, 243; E. H. Moore, Jr. 249; H. B. Newson 229, 232, 233; H. R. Scott 235; D. W. Spence 229; T. U. Taylor 229, 232, 233; W. M. Thornton 172; W. O. Whitescarver 208.

172

FIND the deflection of a homogeneous elastic beam of length $2v$, loaded uniformly, and supported at two points distant u from its middle point.

[*W. M. Thornton.*]

SOLUTION.

The bending moment for load w per inch is, at x inches from midspan,

$$M = \tfrac{1}{2} w(v^2 - 2uv + x^2);$$

whence for the differential equation to the elastica we have

$$EI \frac{d^2y}{dx^2} = \tfrac{1}{2} w(v^2 - 2uv + x^2).$$

The first integration gives

$$EI \frac{dy}{dx} = \tfrac{1}{2} w(v^2x - 2uvx + \tfrac{1}{3} x^3),$$

no constant being added because the declivity of the elastica is zero at midspan. The second integration gives

$$EIy = \tfrac{1}{2} w \left(\frac{v^2x^2}{2} - uvx^2 + \tfrac{1}{12} x^4 \right).$$

The deflection is found by making $x = u$;

$$\therefore\ d = \tfrac{1}{24} \frac{w}{EI} (u^4 - 12u^3v + 6u^2v^2).$$

[*W. M. Thornton.*]

208

PROVE that the surface of an oblate spheroid whose major semiaxis is a and eccentricity e, is equal to

$$2a^2\pi\left[1 + \text{Naperian log}\left(\frac{1+e}{1-e}\right)^{\frac{1-e^2}{2e}}\right].$$

[*R. S. Woodward.*]

SOLUTION.

The surface of the spheroid generated by $a^2y^2 + b^2x^2 = a^2b^2$ when it is rotated about its minor axis is 2π times

$$\int x\,ds = \int \sqrt{x^2 + \frac{a^4}{b^2}y^2}.dy = \int \frac{a^2}{b^2}\sqrt{(1-e^2) + e^2y^2}.dy.$$

From this, by the ordinary integration-process, the result follows.

[*W. O. Whitescarver; Artemas Martin; J. E. Hendricks.*]

215

Two equal circles, radii r, intersect; find the average area common to both. [*Artemas Martin.*]

SOLUTION.

Let A and B be the centres of the two circles; C and E the intersections of their circumferences with one another; F and G their intersections with AB; D the intersection of AB and CE.

While the centre A is fixed, the centre B will be at the distance $2x$ from A if it is anywhere on the circumference described from centre A with radius $AB = 2x$. Hence, for every value of x, B can have $4\pi x$ positions.

Now $CD = \sqrt{(r^2 - x^2)}$, arc $CE = 2r\cos^{-1}\left(\frac{x}{r}\right)$; sector $BCFE = r^2\cos^{-1}\left(\frac{x}{r}\right)$, triangle $CBE = x\sqrt{(r^2 - x^2)}$, segment $CFED = r^2\cos^{-1}\left(\frac{x}{r}\right) - x\sqrt{(r^2 - x^2)}$, and the area common to both circles is

$$2r^2\cos^{-1}\left(\frac{x}{r}\right) - 2x\sqrt{(r^2 - x^2)} = S.$$

The *average* area required is

$$A = \int_0^r S \times 4\pi x\,dx \div \int_0^r 4\pi x\,dx,$$

$$= 4\int_0^r x\cos^{-1}\left(\frac{x}{r}\right)dx - \frac{4}{r^2}\int_0^r x^2\sqrt{(r^2 - x^2)}\,dx,$$

$$= \tfrac{1}{4}\pi r^2.$$

[*Artemas Martin.*]

216

Two equal spheres, radii r, intersect; find the average volume common to both. [*Artemas Martin.*]

SOLUTION.

Let $2x = AB =$ the distance between the centres of the spheres.

While the centre A is fixed, the centre B will be at the distance $2x$ from A if it is anywhere on the surface of the sphere whose centre is at A, and radius $AB = 2x$. Hence, for every value of x, B can have $16\pi x^2$ positions.

Now

$$CD = \sqrt{(r^2 - x^2)}, \quad FD = DG = r - x;$$

volume of segment

$$CGED = \tfrac{1}{2}(r - x) \times \pi(r^2 - x^2) + \tfrac{1}{6}\pi(r - x)^3,$$

and volume $CFEGC$, the volume common to both spheres,

$$= \pi(r - x)(r^2 - x^2) + \tfrac{1}{3}\pi(r - x)^3 = V.$$

The *average* volume required is

$$A = \int_0^r V \times 16\pi x^2 dx \div \int_0^r 16\pi x^2 dx,$$

$$= \frac{\pi}{r^3}\int_0^r [3(r - x)(r^2 - x^2) + (r - x)^3]x^2 dx,$$

$$= \frac{\pi}{r^3}\int_0^r (4r^3 - 6r^2x + 2x^3)x^2 dx,$$

$$= \tfrac{1}{6}\pi r^3. \qquad [\textit{Artemas Martin.}]$$

218

THE length of none of three lines exceeds a; find the probability that an acute triangle can be formed with them. [*Artemas Martin.*]

SOLUTION.

Let x, y, and z denote the lengths of the lines, x being the greatest and z the least.

In order that the triangle may be acute, we must have $z > \sqrt{(x^2 - y^2)}$,

$$\therefore \quad p = \int_0^a \int_0^x \int_{\sqrt{(x^2-y^2)}}^x dxdydz \div \int_0^a \int_0^x \int_0^x dxdydz,$$

$$= \frac{3}{a^3} \int_0^a \int_0^x \int_{\sqrt{(x^2-y^2)}}^x dxdydz,$$

$$= \frac{3}{a^3} \int_0^a \int_0^x [x - \sqrt{(x^2-y^2)}]\, dxdy,$$

$$= \frac{3}{a^3} \int_0^a (1 - \tfrac{1}{4}\pi) x^2 dx = 1 - \tfrac{1}{4}\pi.$$

[*Artemas Martin.*]

229

If $a^2 = b^2 - bc$ and $b^2 = c^2 - ac$, then $c^2 = a^2 - ab$. [*Frank Morley.*]

SOLUTION.

Eliminating a, b, and c, respectively, from the given equations, we obtain

$$(b - c)(b^3 + b^2c - 2bc^2 - c^3) = 0, \tag{1}$$

$$a(c^3 + c^2a - 2ca^2 - a^3) = 0, \tag{2}$$

$$a(a^3 + a^2b - 2ab^2 - b^3) = 0. \tag{3}$$

The symmetry of the right hand factors show that $c^2 = a^2 - ab$ is satisfied for three values of (1), (2), and (3), and also, for $a = b = c = 0$; but for $a = 0$, $b = c =$ a finite quantity, the given equations are satisfied, but $c^2 = a^2 - ab$ is not satisfied. The proposition is true for every other case, but fails in this particular one. [*E. Frisby.*]

230

Solve the equations

$$x^2(y - x) = a,$$

$$y^2(z - x) = b,$$

$$z^2(x - y) = c.$$ [*Frank Morley.*]

SOLUTION.

Let $y = vx$, $z = wx$; then the three given equations become

$$x^3(v - w) = a,$$

$$v^2x^3(w - 1) = b,$$

$$w^2x^3(v - 1) = -c.$$

$$\therefore \quad \frac{v^2(w - 1)}{v - w} = \frac{b}{a},$$

$$\frac{w^2(v - 1)}{v - w} = -\frac{c}{a}.$$

Subtracting the latter from the former, we have

$$vw - (v + w) = \frac{b + c}{a}.$$

Substituting this value of vw in either of the two equations just written, we have

$$(a + c)v + (a + b)w = -(b + c),$$

or
$$w = -\frac{(a + c)v + (b + c)}{a + b}.$$

$$\therefore \quad v^2 + \frac{2b}{a + c}v = -\frac{b(b + c)}{a(a + c)},$$

$$v = -\frac{b}{a + c} \pm \frac{1}{a + c}\sqrt{-\frac{bc}{a}(a + b + c)};$$

$$\therefore \quad w = -\frac{c}{a + b} \mp \frac{1}{a + b}\sqrt{-\frac{bc}{a}(a + b + c)}.$$

From the first of the three given equations, we obtain

$$x = \left(\frac{a}{v - w}\right)^{\frac{1}{3}};$$

$$\therefore \quad x = \left(\frac{a}{-\frac{b}{a + c} + \frac{c}{a + b} \pm \left(\frac{1}{a + c} + \frac{1}{a + b}\right)\sqrt{-\frac{bc}{a}(a + b + c)}}\right)^{\frac{1}{3}}.$$

[*R. A. Harris.*]

231

IF we take products of n consecutive terms of the arithmetical series a, $a - d$, etc., commencing for the first product with the first term, for the second product with the second term, and so on; and then multiply these products by the coefficients in the expansion of $(1 - x)^n$, the aggregate will be $n!\, d^n$, which is independent of the first term. [*W. W. Johnson.*]

SOLUTION.

1. For any series $a_0, a_1, a_2, \ldots$, the first term in the nth order of differences is equal to

$$a_n - na_{n-1} + \frac{n(n-1)}{1.2} a_{n-2} + \ldots + (-1)^n a_0.$$

2. If a_r be the rational function $Ar^n + Br^{n-1} + \ldots$, it is easy to see that the terms of the nth order of differences will be independent of r, and each equal to $n!\, A$.

3. The series formed as stated in question will have its $(r+1)$th term equal to

$$(a - rd)(a - \overline{r+1}.d) \ldots (a - \overline{r+n-1}.d),$$

which is of the form $(-d)^n.r^n + \ldots$; whence, by (1) and (2), we have

$$a_n - na_{n-1} + \ldots + (-1)^n a_0 = n!\,(-d)^n;$$

and, dividing by $(-1)^n$, we get the required result, viz.:

$$a_0 - na_1 + \frac{n(n-1)}{1.2} a_2 + \ldots + (-1)^n a_n = n!\, d^n,$$

wherein $a_0, a_1, a_2, \ldots a_n$ are any $n+1$ consecutive terms of the series.

4. It may be added that if the coefficients of $(1 - x)^m$ be applied to any $m+1$ consecutive terms of the series, the aggregate will be *zero* when $m > n$; and similarly for any series in which a_r is a rational nth-degree function of r.

[*James McMahon.*]

232 and 233

IF a square be inscribed in the face of a cube, the plane determined by one side and the corner of the opposite face corresponding to the adjacent corner of the same face touches the inscribed sphere. [*T. M. Blakslee.*]

SOLUTION.

Take the centre of the cube as origin of co-ordinates. The plane required passes through the points

$$(a, 0, a);\quad (0, a, a);\quad (a, a, -a),$$

and hence has for equation

$$2x + 2y + z = 3a,$$

and for distance from the origin $r = a$.

There are twenty-four such planes. [*T. U. Taylor; H. B. Newson.*]

236

If p, q, r, s are the lengths, supposed unequal, of the sides of a quadrilateral, prove that

$$[(p+q)(p+r)(p+s)(q+r)(q+s)(r+s)]^2$$
$$> [(p+q+r-s)(p+q-r+s)(p-q+r+s)(-p+q+r+s)]^2.$$

[*R. H. Graves.*]

The restrictions are altogether unnecessary. p, q, r, s may be any numbers whatever, positive, negative, or even 0, provided one at least is different from the others; and even when they are all equal, it is only a limiting case. If the sign $>$ means "not less than," the proposition is absolutely universal.

PROOF.

$$(p+q)^2 > (p+q)^2 - (r-s)^2, \text{ or } > (p+q+r-s)(p+q-r+s);$$

$$(r+s)^2 > (r+s)^2 - (p-q)^2, \text{ or } > (r+s+p-q)(r+s-p+q).$$

Forming the other squares in the same way, and multiplying them together, the proposition results. [*E. Frisby.*]

237

The axes of an ellipse are given, and one focal distance of a point on the curve. Find the ordinate of the point drawn to the major axis.

[*O. L. Mathiot.*]

SOLUTION.

$$r = \frac{b^2}{a + \sqrt{(a^2 - b^2)}.\cos v}, \quad y = r \sin v;$$

or

$$\sqrt{a^2 - b^2}.\, r \cos v = b^2 - ar,$$

$$\sqrt{a^2 - b^2}.\, r \sin v = \sqrt{a^2 - b^2}.\, y;$$

whence, by squaring and adding, we have

$$(a^2 - b^2)r^2 = (a^2 - b^2)y^2 + b^4 - 2ab^2r + a^2r^2,$$

$$y^2 = \frac{2ab^2r - b^2(b^2 + r^2)}{a^2 - b^2}. \qquad [E.\ Frisby.]$$

238

LET P be a point on an equilateral hyperbola. Find the locus of the point of intersection of the ordinate of P and the perpendicular bisector of the longer of the supplemental chords drawn to P. [*R. H. Graves.*]

SOLUTION.

It can be easily proved that a rhomb may be constructed having for one diagonal the chord and the ends of the other diagonal on the axes. The equation to this diagonal (the perpendicular bisector) is evidently

$$\frac{x}{x' - a} + \frac{y}{y'} = 1, \text{ where } x'^2 - y'^2 = a^2.$$

Eliminate from these equations and $x = x'$ the co-ordinates x' and y'. The equation to the required locus is thus found to be

$$(x - a)^2y^2 - a^2x^2 + a^4 = 0. \qquad [R.\ H.\ Graves.]$$

239

FIND the equation to QR of Exercise 205, P being given.

[*R. H. Graves.*]

SOLUTION.

Let the equation to the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and let (x', y') be the co-ordinates of the point P.

QR passes through $\left(\frac{a^4}{(a^2 - b^2)x'}, \frac{-b^4}{(a^2 - b^2)y'}\right)$. (See solution to Exercise 205.) QR also passes through the point $\left(-\frac{a^2}{x'}, 0\right)$. For, "If two chords of a

conic be drawn through two points on a diameter equidistant from the centre, any conic through the extremities of these chords will be cut by that diameter in points equidistant from the centre." (See Smith's Conic Sections, p. 202.)

Therefore the equation to the line QR is

$$\frac{xx'}{a^2} + \frac{yy'}{kb^2} = -1 \text{ (1), where } k = \frac{1-e^2}{1+e^2}.$$

COR. (1) touches $\frac{x^2}{a^2} + \frac{y^2}{k^2b^2} = 1$ at $(-x', -ky')$.

This is a special case of Ex. 3 of Art. 272, Salmon's Conic Sections.

REM. A similar investigation would apply to the hyperbola.

[*R. H. Graves.*]

241

AN ellipse cuts a confocal lemniscate where its ordinate is a maximum. Find the eccentricity of the ellipse. [*R. H. Graves.*]

SOLUTION.

Use the equations to the curve in vectorial co-ordinates,

$$r + r' = 2a \text{ (1)} \quad \text{and} \quad rr' = a^2e^2 \text{ (2)}.$$

At the points of intersection of (1) and (2)

$$r = a + b \quad \text{and} \quad r' = a - b.$$

When the ordinate is a maximum it is evident that r and r' are at right angles. Then,

$$4a^2e^2 = (a+b)^2 + (a-b)^2;$$

$$\therefore \quad a^2 = 3b^2, \quad \text{or} \quad e = \sqrt{\tfrac{2}{3}}. \qquad [R.\ H.\ Graves.]$$

243

A NODAL quartic passes through the twelve intersections of three conics. Show that the polars of the node with regard to the conics meet in a point.

[*Frank Morley.*]

SOLUTION.

Any quartic passing through the twelve intersections of the conics $u = 0$, $v = 0$, $w = 0$, can be represented by the equation

$$fvw + guw + huv = 0,$$

for two other conditions would fix the quartic, and would be just sufficient to determine $f : g : h$.

If the quartic be nodal, the co-ordinates x, y, z of the node satisfy the three derivatives as to x, y, z; then, eliminating f, g, h, the equation to the locus of the node is

$$\begin{vmatrix} v_1w + vw_1 & u_1w + uw_1 & u_1v + uv_1 \\ v_2w + vw_2 & u_2w + uw_2 & u_2v + uv_2 \\ v_3w + vw_3 & u_3w + uw_3 & u_3v + uv_3 \end{vmatrix} = 0,$$

wherein the subscripts denote differentiation as to x, y, z, respectively. This equation easily reduces to

$$\begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} = 0,$$

which represents the Jacobian of the three conics, and is evidently the locus of a point whose three polars $xu_1 + yu_2 + zu_3 = 0$, etc. meet in a common point. Hence the theorem. [*James McMahon.*]

244

If an oval of Cassini, confocal with a meridian of the earth, cuts it at a certain point, the angle at which the curves intersect is equal to the reduction of latitude at that point. [*R. H. Graves.*]

SOLUTION.

The focal radii at the point make equal angles with the central radius and the normal of the oval*; and they are equally inclined to the normal to the meridian. Hence the truth of the proposition. [*R. H. Graves.*]

245

The probability that an event A happens is p_1; the probability that an event B happens is p_2; the probability that neither happens is p_3; required, the probability that both happen. [*L. M. Hoskins.*]

SOLUTION.

Regarding the occurrence of the events A and B there are only four possible cases:—

1. A and B both happen;
2. A happens but not B;
3. B happens but not A;
4. Neither A nor B happens.

*See Williamson's Diff. Calc., Art. 193, Ex. 2.

Let a, b, c, d represent the probabilities of these several cases. The four cases being *mutually exclusive*, we must have

$$a + b + c + d = 1.$$

Also, each of the first two cases involves the occurrence of A, hence the probability of A occurring is $a + b$. In the same way the probability of B occurring is $a + c$. The data of the problem, then, are

$$\begin{aligned} a + b + c + d &= 1, \\ a + b &= p_1, \\ a + c &= p_2, \\ d &= p_3; \end{aligned}$$

and we are to find a. The result is $a = p_1 + p_2 + p_3 - 1$.

[*L. M. Hoskins.*]

249

Cut the two edges AB, CD of the tetraedron $ABCD$ by the plane Π in P, Q respectively; and take P', Q' the harmonic conjugates to P, Q relative to AB, CD. Draw a plane Σ through $P'Q'$ and let M, N be the points in which it cuts AC, BD. Then will the join MN intersect both PQ and $P'Q'$; and in points R, R' which divide MN harmonically. [*E. H. Moore, Jr.*]

SOLUTION.

In fact, the three lines AC, BD, PQ determine an hyperboloid of one sheet, of which one set of generators is the series of lines lying across these three lines, which belong to the second set of generators. Every line of one set intersects every line of the other set, and the lines of one set intersect the lines of the other set in projective ranges of points. ABP, CDQ are evidently lines of the first set, and the line of the second set crossing ABP at P', the fourth harmonic point, would cross CDQ at the fourth harmonic point Q'; that is, $P'Q'$ is a generator of the second set, and every line crossing AC, BD, PQ crosses $P'Q'$ also, and the four points of intersection make an harmonic range; or, what comes to the same thing, every line MN (lying in a plane Σ through and, therefore,) intersecting $P'Q'$ at R', and also intersecting AC, BD at M, N, intersects also PQ at, say, R, and the four points $MNRR'$ make an harmonic range.

To obtain Ex. 207, let Π be the plain at infinity, when the fourth harmonic point to any two points and the point of intersection of the join-line with the plane Π will be simply the middle point of the segment joining the two points.

[*E. H. Moore, Jr.*]

250

A HOMOGENEOUS sphere rests on another such sphere of equal mass, which rests on a table. Everything being smooth and the system being slightly shaken, show that the spheres will separate when the upper one has turned through the angle $\cos^{-1}(1/3 - 1)$. [*Frank Morley.*]

SOLUTION.

Take the origin at the centre of the lower sphere at the beginning of motion, and a horizontal line through this centre in the plane of motion for axis of x. Let x, y; $x', 0$ be the co-ordinates of the centres of the spheres, φ the angle the line joining the centres makes with the vertical, R the reaction between the spheres, m the mass of a sphere, and a its radius. The equations of motion in x are

$$m\frac{d^2x}{dt^2} = R\sin\varphi, \qquad m\frac{d^2x'}{dt^2} = -R\sin\varphi.$$

From these equations we have $x = -x'$. The positions of the spheres furnish the relations

$$x = a\sin\varphi, \qquad y = 2a\cos\varphi,$$

$$x' = -a\sin\varphi.$$

The values of the derivatives, $\frac{dx}{dt}$, $\frac{dx'}{dt}$, $\frac{dy}{dt}$, and that of the *force function*, $2agm(1 - \cos\varphi)$, give, by the principle of *vis viva*,

$$(2 - \cos^2\varphi)\, a^2\frac{d\varphi^2}{dt^2} = 2ag(1 - \cos\varphi).$$

Resolving the forces along the line joining the centres of the spheres, since $2a$ is constant, we have,

$$2am\frac{d\varphi^2}{dt^2} = gm\cos\varphi - R.$$

At the instant the spheres separate $R = 0$; and substituting the value of $a^2\frac{d\varphi^2}{dt^2}$ in the equation of *vis viva*, the result is

$$\cos^3\varphi - 6\cos\varphi + 4 = 0.$$

The roots of this equation are

$$+\sqrt{3} - 1, \qquad -\sqrt{3} - 1, \qquad +2.$$

Only the first root can be used, and hence the result. If the lower sphere were fixed the spheres would separate when $\cos\varphi = \frac{2}{3}$. [*A. Hall.*]

EXERCISES.

263

INTEGRATE $\frac{(x-1)\,xdx}{x^3+x^2+x+1}$, from $x=0$ to $x=1$. [*O. Root, Jr.*]

264

INTEGRATE $\frac{d\varphi}{1-(1-3\tan^2\varphi)}$, from $\varphi=0$ to $\varphi=\frac{1}{4}\pi$. [*O. Root, Jr.*]

265

INTEGRATE the equation

$$dx=\tfrac{1}{2}\sqrt{[a-\tfrac{1}{2}(b^2+x^2)]}\,dy. \qquad [\textit{Artemas Martin.}]$$

266

IT is required to find nine biquadrate numbers whose sum is a square number. [*Artemas Martin.*]

267

EXAMINE for asymptotes $\rho^2=a^2\frac{\sin 3\theta}{\cos\theta}$. [*O. Root, Jr.*]

268

FIND the maximum rectangle inscribed in the witch $y^2+a^2=\frac{a^3}{x}$.

[*O. Root, Jr.*]

269

SUPPOSE a series of ellipses described on the same conjugate axis. Find the eccentricity of that one for which the distance from the centre to the directrix is a minimum. [*H. B. Newson.*]

270

GIVEN a series of ellipses described on the same transverse axis. Find the eccentricity of that one for which the first pedal curve passes from a smooth oval form to an indented form having four points of inflection.

[*H. B. Newson.*]

CONTENTS.

	PAGE.
On the Eccentricity of Plane Sections of Quadrics. By H. B. NEWSON, .	1
The Problem of the Planimeter. By W. H. ECHOLS, ...	9
The Approximate Inscription of Certain Regular Polygons. By H. A. HOWE, ...	12
Note on Perspective Projection. By W. H. ECHOLS, ...	15
On the Expression of the Hessian of a Binary Quantic in Terms of the Roots. By JAMES MCMAHON, ...	17
Note on Symbols. By A. HALL, ...	19
Solutions of Exercises 172, 208, 215, 216, 218, 229–233, 236–239, 241, 243–245, 549, 250, ...	20
Exercises 263–270, ...	32

ANNALS OF MATHEMATICS.

Terms of subscription: $2 for a volume of at least 200 pages; THE MONEY MUST IN ALL CASES ACCOMPANY THE ORDER. All drafts and money orders should be made payable to the order of ORMOND STONE, University of Virginia, Va, U. S. A.

SCIENTIFIC AMERICAN
ESTABLISHED 1845.

Is the oldest and most popular scientific and mechanical paper published and has the largest circulation of any paper of its class in the world. Fully illustrated. Best class of Wood Engravings. Published weekly. Send for specimen copy. Price $3 a year. Four months' trial, $1. MUNN & CO., PUBLISHERS, 361 Broadway, N.Y.

ARCHITECTS & BUILDERS
Edition of Scientific American.

A great success. Each issue contains colored lithographic plates of country and city residences or public buildings. Numerous engravings and full plans and specifications for the use of such as contemplate building. Price $2.50 a year, 25 cts. a copy. MUNN & CO., PUBLISHERS.

PATENTS may be secured by applying to MUNN & Co., who have had over 40 years' experience and have made over 100,000 applications for American and Foreign patents. Send for Handbook. Correspondence strictly confidential.

TRADE MARKS.

In case your mark is not registered in the Patent Office, apply to MUNN & Co., and procure immediate protection. Send for Handbook.

COPYRIGHTS for books, charts, maps, etc., quickly procured. Address

MUNN & CO., Patent Solicitors,
GENERAL OFFICE: 361 BROADWAY, N. Y

PUBLICATIONS RECEIVED.

Untersuchungen über das Cometensystem 1843 I, 1880 I, and 1882 II. I. THEIL. Der grosse September-comet 1882 II. Von DR. HEINRICH KREUTZ. (Publication der Sternwarte in Kiel.)

A Comparison of the Electric Theory of Light and Sir William Thomson's Theory of a Quasi-Sabile Ether. By J. WILLARD GIBBS. (American Journal of Science.)

On the Form and Position of the Sea Level with Special Reference to its Dependence on Superficial Masses Symmetrically Disposed about a Normal to the Earth's Surface. By ROBERT SIMPSON WOODWARD. (United States Geological Survey.)

Double-Star Observations made at the Lick Observatory. By S. W. BURNHAM. (Astronomische Nachrichten.)

Ephemeris for Physical Observations of the Moon.—Ephemeris of the Satellites of Uranus, 1889. By A. MARTH. (Monthly Notices of the Royal Astronomical Society.)

Proceedings of the Tenth Annual Meeting of the Polytechnic Society of Kentucky.

Some Modifications of the Methods of Organic Analysis by Combustion. By WM. L. DUDLEY.

The Williams College Catalogue of North Polar Stars, Right Ascensions for 1885 0. By TRUMAN HENRY SAFFORD.

Cours d'Arithmètique. Par A. TARTINVILLE.

Rapport Annuel sur l'Etat de l'Observatoire de Paris, pour l'Année 1888. Par M. le Contre-Amiral MOUCHEZ.

Differential Method of Computing Apparent Places of Stars for Latitude Work. By ERASMUS D. PRESTON. (Journal of the Franklin Institute.)

Geometric Methods of Obtaining the Equation of a Tangent to a Conic. By H. J. MESSENGER.

THE ANALYST.

The undersigned has a few complete sets (unbound) of *The Analyst* yet on hand which he will send to any address in the United States, free of charges for transportation, for $15 per set of ten volumes; or he will send any one, or all of the last six volumes for $1 per volume; or he will send any one or more Nos. of the last six volumes for 20 cents per No. Address J. E. HENDRICKS, 1400 Court Ave., Des Moines, Iowa.

R. BERESFORD, Printer, 617 E Street, Washington, D. C.